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論 文

On Torus Knots obtained by Self-Fusions on the Torus

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Abstract

For the torus knot on the standard torus in S^3, we can get another torus knot by performing a self-fusion on the torus. In the present paper, we detect the relationship between those two torus knots, and show a necessary and sufficient condition for the torus knot to become the trivial knot by once self-fusion. Moreover for a given positive integer $p > 0$, we show a necessary and sufficient condition for $T(p, q)$ to become the trivial knot by maximal times of self-fusions.

Keywords: Torus knots, self-fusions.

1 Introduction

Let $V = S^1 \times D^2$ be a standard solid torus in S^3, and put $\ell = S^1 \times \{x\}$ and $m = \{y\} \times \partial D^2$, where x is a point in ∂D^2 and y is a point in S^1. Then ℓ is called a longitude of V and m is called a meridian of V (Figure 1). Let C be a simple closed curve in ∂V such that C intersects m in p points with the same directions and intersects ℓ in q points with the same directions. Then C is called a torus knot of type (p, q) and denoted by $T(p, q)$ (Figure 1).

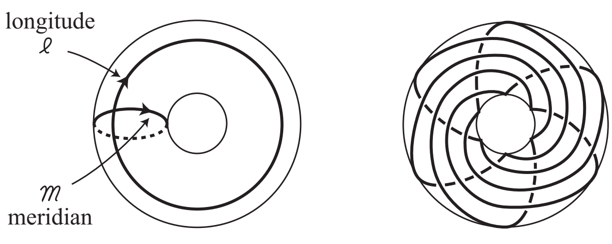

Figure 1: Torus and a torus knot.

In general, for two integers $p \geq 0,\ q \geq 0$, we can take such a simple closed curve or a family of simple closed curves. Then, the following is the fundamental fact and we omit the proof.

Fact 1.1 ([1]) $T(p, q)$ *is a simple closed curve if and only if* $\gcd(p, q) = 1$.

For a convenient method to draw torus knots, see [2], and for a method using 3-dimensional coordinate, see [3].

Let α be an arc in ∂V with $\alpha \cap T(p,q) = \partial\alpha$, and is not parallel to a subarc of $T(p,q)$. Then we can perform a self-fusion along α on the torus as illustrated in Figure 2.

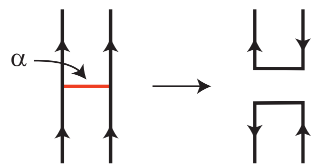

Figure 2: Self-fusion along α.

By this operation, we can get another torus knot denoted by $T(r,s)$, and some examples are illustrated in Figures 3 and 4.

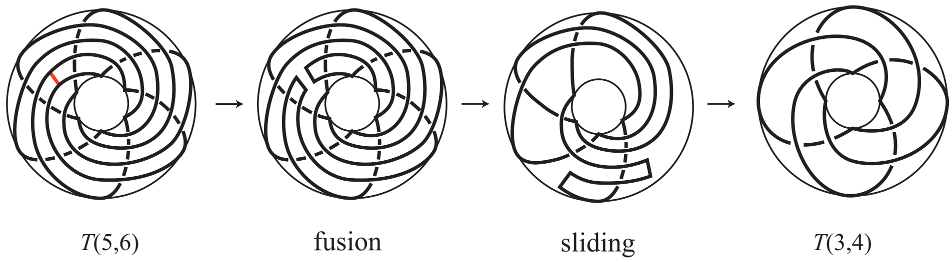

Figure 3: $T(5,6) \to T(3,4)$.

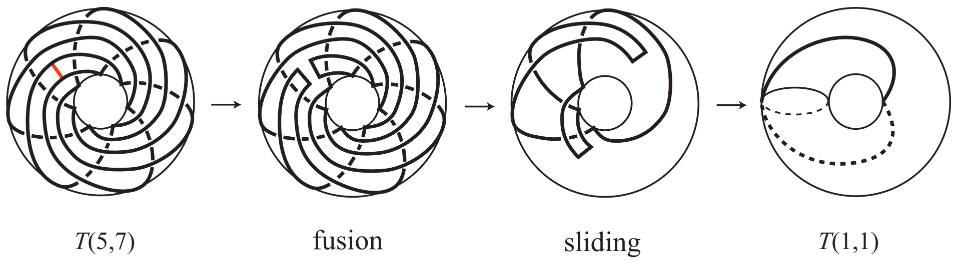

Figure 4: $T(5,7) \to T(1,1)$.

The first basic result is the following:

Theorem 1.2 *Let $p \geq 2$, $q \geq 1$ be two integers with $\gcd(p,q) = 1$. Then $T(r,s)$ is obtained from $T(p,q)$ by once self-fusion if and only if $|ps - qr| = 2$, $0 \leq r \leq p-2$, $0 \leq s \leq q$ and $\gcd(r,s) = 1$.*

By the above theorem, we have the following corollary and proposition:

Corollary 1.3 *A necessary and sufficient condition to get a trivial knot from a non-trivial $T(p,q)$ by once self-fusion is that $p \equiv \pm 2 \pmod{q}$ or $q \equiv \pm 2 \pmod{p}$.*

Proposition 1.4 *For a given $p \geq 2$, a necessary and sufficient condition to get a trivial knot from a non-trivial $T(p,q)$ by maximal times of self-fusions is that $q \equiv \pm 1 \pmod{p}$. In this case, the number of times of self-fusions is as follows:*

$$\text{The number of times} = \begin{cases} \frac{p-1}{2} & (p \textit{ is odd}). \\ \frac{p}{2} & (p \textit{ is even and } q \neq p-1). \\ \frac{p-2}{2} & (p \textit{ is even and } q = p-1). \end{cases}$$

Examples Put $p = 13$, then the sequence to get a trivial knot from $T(p, q)$ by self-fusions for $q = 12, 11, \ldots, 2$ is the following:

(1) $q = 12$: $T(13, 12) \to T(11, 10) \to T(9, 8) \to T(7, 6) \to T(5, 4) \to T(3, 2) \to T(1, 0)$.

(2) $q = 11$: $T(13, 11) \to T(1, 1)$.

(3) $q = 10$: $T(13, 10) \to T(5, 4) \to T(3, 2) \to T(1, 0)$.

(4) $q = 9$: $T(13, 9) \to T(7, 5) \to T(1, 1)$.

(5) $q = 8$: $T(13, 8) \to T(3, 2) \to T(1, 0)$.

(6) $q = 7$: $T(13, 7) \to T(9, 5) \to T(5, 3) \to T(1, 1)$.

(7) $q = 6$: $T(13, 6) \to T(9, 4) \to T(5, 2) \to T(1, 0)$.

(8) $q = 5$: $T(13, 5) \to T(3, 1)$.

(9) $q = 4$: $T(13, 4) \to T(7, 2) \to T(1, 0)$.

(10) $q = 3$: $T(13, 3) \to T(5, 1)$.

(11) $q = 2$: $T(13, 2) \to T(1, 0)$.

As the next step of Corollary 1.3, we can put the following problem 1:

Problem 1 Find a necessary and sufficient condition to get a trivial knot from a $T(p, q)$ by exactly twice self-fusions.

More generally, we can put the following problem 2:

Problem 2 For a given $T(p, q)$, determine the number of times of self-fusions to get a trivial knot by sequence of self-fusions.

2 Proofs of Theorem 1.2, Corollary 1.3 and Proposition 1.4

Proof of Theorem 1.2. Suppose $T(r, s)$ is obtained from $T(p, q)$ by once self-fusion. Figure 5 shows a self-fusion on $T(7, 4)$. Note that Figure 5 is illustrated as a torus braid.

Put the numbers $0, 1, 2, \ldots, p-1$ at the both ends of the braid. Then after the self-fusion, put the arrows on each strand with the red starting point.

Then, up-arrows and down-arrows appear along the following sequences:

$$\uparrow: 0 \to 4 \to 1. \qquad \downarrow: 0 \to 3 \to 6 \to 2 \to 5 \to 1.$$

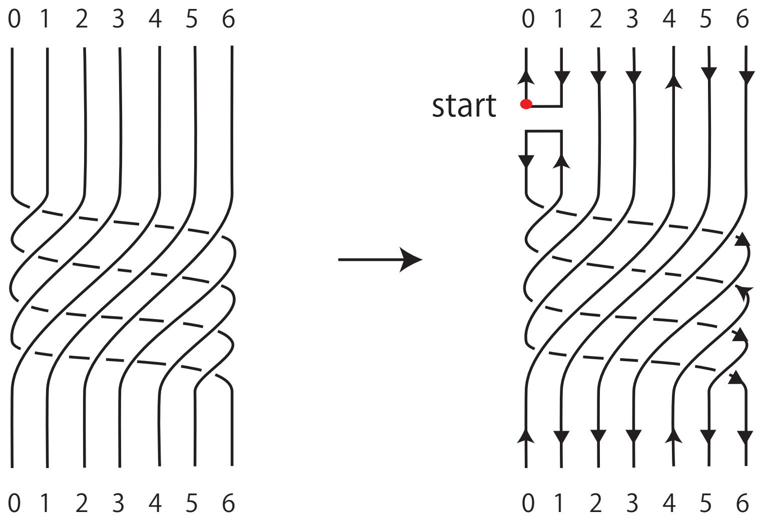

Figure 5: Self-fusion on $T(7,4)$.

Since these numbers are counted under modulo $p = 7$, we can recognize as follows:

$$\uparrow\!:\ 0 \to 4 \to 8. \qquad\qquad \downarrow\!:\ 0 \to -4 \to -8 \to -12 \to -16 \to -20.$$

In general, these sequences show the following arithmetic progressions:

$$\uparrow\!:\ 0 \to q \to 2q \to \cdots \to nq. \qquad \downarrow\!:\ 0 \to -q \to -2q \to \cdots \to -mq.$$

Since these sequences end up with 1, we can put $nq \equiv 1 \pmod p$ and $mq \equiv -1 \pmod p$.

By the way, so far we have considered the longitudinal direction, but we can next consider the meridional direction. Then by the similar arguments, we have the following sequences:

$$\uparrow\!:\ 0 \to p \to 2p \to \cdots \to xp. \qquad \downarrow\!:\ 0 \to -p \to -2p \to \cdots \to -yp.$$

Then we have the equations: $xp \equiv 1 \pmod q$ and $yp \equiv -1 \pmod q$ similarly. Then summarizing these equations, we have the following. Here, n, m, x and y are the minimal positive integers satisfying the following equations (1) and (2).

$$\begin{cases} nq & \equiv & 1 \pmod p. \\ mq & \equiv & -1 \pmod p. \end{cases} \quad (1) \qquad \begin{cases} xp & \equiv & 1 \pmod q. \\ yp & \equiv & -1 \pmod q. \end{cases} \quad (2)$$

Since r is the number of the longitudinal directions and s is the number of the meridional directions, we have the following:

$$r = |n - m| \quad \text{and} \quad s = |x - y|. \tag{3}$$

Now, by Eq. (1), there are positive integers $k,\ \ell$ satisfying the following Eq. (4):

$$\begin{cases} nq &= kp+1. \\ mq &= lp-1. \end{cases} \tag{4}$$

Since $n,\ m$ are minimal, $k,\ \ell$ are minimal too, and satisfy the following Eq. (5):

$$\begin{cases} kp &\equiv -1 \pmod{q}. \\ \ell p &\equiv 1 \pmod{q}. \end{cases} \tag{5}$$

By Eq. (2) and Eq. (5), and by the minimality of x, y, k, ℓ, we have $x = \ell$ and $y = k$. Then by substituting these equations for Eq. (4), we have the following Eq. (6):

$$\begin{cases} nq &= yp+1. \\ mq &= xp-1. \end{cases} \tag{6}$$

By subtracting the both sides of Eq. (6), we have the following Eq. (7):

$$(n-m)q = (y-x)p + 2. \tag{7}$$

Here, we will show $(n-m)(y-x) \geq 0$. In the following, we use the method similar to the proof of Proposition 2.1 of [4].

(i) Suppose $n - m > 0$.

In this case, by $(n-m)q > 0$ and by Eq. (7), we have $(y-x)p + 2 > 0$.

If $y - x < 0$, then $p > 1$ implies $(y-x)p \leq -2$. This is a contradiction.

Hence $y - x \geq 0$ and $(n-m)(y-x) \geq 0$.

(ii) Suppose $n - m \leq 0$.

In this case, by $(n-m)q \leq 0$ and by Eq. (7), we have $(y-x)p + 2 \leq 0$.

If $y - x > 0$, then $p > 1$ implies $(y-x)p \geq 2$. This is a contradiction.

Hence $y - x \leq 0$ and $(n-m)(y-x) \geq 0$.

By (i) and (ii), we have $(n-m)(y-x) \geq 0$.

This means that $n - m$ and $y - x$ are 0 or have the same signs. Then we have the following:

$$\begin{aligned} ps - qr &= p|x-y| - q|n-m| \text{ (by Eq. (3))} \\ &= p|y-x| - q|n-m| \\ &= \pm\{p(y-x) - q(n-m)\} \text{ (by the same signs)} \\ &= \pm 2 \text{ (by Eq. (7))}. \end{aligned}$$

Hence we get $ps - qr = \pm 2$.

Moreover, since the self-fusion is an operation to fuse the adjacent two strands, p decreases by at least two. This means $0 \leq r \leq p-2$. In addition, we have $0 \leq s \leq q$ because q does not increase, and by Fact 1.1, we have $\gcd(r, s) = 1$. This completes the proof of the first half.

Conversely, suppose (r, s) satisfies that $ps - qr = \pm 2,\ 0 \leq r \leq p-2,\ 0 \leq s \leq q$ and $\gcd(r, s) = 1$. Then, in the following, we show that such pair (r, s) is unique.

So suppose there is another pair (r', s') with $ps' - qr' = \pm 2,\ 0 \leq r' \leq p-2,\ 0 \leq s' \leq q$ and $\gcd(r', s') = 1$. Then we have the following two cases:

$$(\mathrm{I}) \begin{cases} ps - qr &= \pm 2. \\ ps' - qr' &= \pm 2. \end{cases} \qquad (\mathrm{II}) \begin{cases} ps - qr &= \pm 2. \\ ps' - qr' &= \mp 2. \end{cases}$$

First suppose we are in case (I).

By subtracting the both sides, we have $p(s-s') - q(r-r') = 0$ and hence $p(s-s') = q(r-r')$.

If $r - r' = 0$, then $s - s' = 0$ and $(r, s) = (r', s')$.

If $r - r' \neq 0$, then $\frac{s-s'}{r-r'} = \frac{q}{p}$.

Since $\frac{q}{p}$ is an irreducible fraction, we can put $r - r' = pk$, $s - s' = qk$ and have $|r - r'| \geq p$.

However, since $0 \leq r \leq p-2$ and $0 \leq r' \leq p-2$, we have $|r - r'| \leq p - 2$. This contradiction shows that $(r, s) = (r', s')$.

Next, suppose we are in case (II).

By adding the both sides, we have $p(s+s') - q(r+r') = 0$ and hence $p(s+s') = q(r+r')$.

If $r + r' = 0$, then $r = r' = s = s' = 0$ and this is a contradiction.

If $r + r' \neq 0$, then $\frac{s+s'}{r+r'} = \frac{q}{p}$.

Since $\frac{q}{p}$ is an irreducible fraction, we can put $r + r' = pk$ and $s + s' = qk$.

Then, since $0 \leq r \leq p-2$, $0 \leq r' \leq p-2$, we have $0 < r + r' < 2p - 4$, and hence $r + r' = p$ and $s + s' = q$. Under this situation, we consider the following two subcases:

(i) p is odd.

If r is even, then s is odd because $\gcd(r, s) = 1$. Then $ps - qr =$ (odd number) $\cdot$ (odd number) $- q\cdot$ (even number) $=$ (odd number).

This contradicts that $ps - qr = \pm 2$.

If r is odd, then r' is even because $r + r' = p$, and s' is odd because $\gcd(r', s') = 1$. Then $ps' - qr' =$ (odd number) $\cdot$ (odd number) $- q\cdot$ (even number) $=$ (odd number).

This contradicts that $ps' - qr' = \mp 2$.

(ii) p is even.

In this case, q is odd because $\gcd(p, q) = 1$.

If s is even, then r is odd because $\gcd(r, s) = 1$. Then $ps - qr =$ (even number) $\cdot$ (even number) $-$ (odd number) $\cdot$ (odd number) $=$ (odd number).

This contradicts that $ps - qr = \pm 2$.

If s is odd, then s' is even because $s + s' = q$, and r' is odd because $\gcd(r', s') = 1$. Then $ps' - qr' =$ (even number) $\cdot$ (even number) $-$ (odd number) $\cdot$ (odd number) $=$ (odd number).

This contradicts that $ps' - qr' = \mp 2$.

After all, these contradictions show that case (II) does not occur.

Therefore, the pair (r, s) is unique and this completes the proof of Theorem 1.2. □

Remark. The above proof of Theorem 1.2 is a proof using number theoretical arguments. However, as an alternative proof, we can prove Theorem 1.2 by using geometric arguments. In fact, we can prove it by noting the intersection number of two torus knots on the same torus.

Proof of Corollary 1.3. Since $T(r, s)$ is a trivial knot if and only if $r = 1$ or $s = 1$ because of $0 \leq r$, $0 \leq s$, we have $|p - qr| = 2$ or $|ps - q| = 2$ by Theorem 1.2. Then $p = qr \pm 2$ or $q = ps \pm 2$, and we have $p \equiv \pm 2 \pmod q$ or $q \equiv \pm 2 \pmod p$. This completes the proof of Corollary 1.3. □

Proof of Proposition 1.4. Let p be a given integer, and suppose $T(r, s)$ is obtained from $T(p, q)$ by a self-fusion. Then since $r \leq p - 2$, to take the maximal times, we need $r = p - 2$. In fact, we have:

Lemma 2.1 *$r = p - 2$ if and only if $q \equiv \pm 1 \pmod p$.*

Proof. Suppose $r = p - 2$. Then $ps - qr = \pm 2 \Longleftrightarrow ps - q(p - 2) = \pm 2 \Longleftrightarrow$
$ps - pq + 2q = \pm 2 \Longleftrightarrow 2q = p(q - s) \pm 2$.

This final equation implies $p(q - s)$ to be even. If p is odd, then we can put $q - s = 2k$ for some k. Then we have $2q = 2pk \pm 2$. This implies $q = pk \pm 1$ and $q \equiv \pm 1 \pmod p$.

If p is even, then q is odd because of $\gcd(p, q) = 1$. Since $r = p - 2$ is even and by $\gcd(r, s) = 1$, s is odd. Then $q - s$ is even and we can put $q - s = 2k$ for some k. This implies $q \equiv \pm 1 \pmod p$ similarly as above.

Conversely suppose $q \equiv \pm 1 \pmod p$. Then we can put $q = pk \pm 1$ for some $k > 0$ because $T(p, q)$ is a non-trivial knot.

If $p > 2$, then $q = pk \pm 1 \geq 2k$ and we can put $q - 2k = s' \geq 0$. Put $r' = p - 2$. Then $ps' - qr' = p(q - 2k) - q(p - 2) = pq - 2pk - qp + 2q = 2q - 2pk = 2(q - pk) = \pm 2$.

If $\gcd(r', s') > 1$, then $\gcd(p - 2, q - 2k) = \gcd(r', s') = 2$ and this implies $\gcd(p, q) = 2$. This contradiction implies that $\gcd(r', s') = 1$ and $T(r', s')$ is obtained from $T(p, q)$ by a self-fusion by Theorem 1.2. Thus we have $(r, s) = (r', s') = (p - 2, q - 2k)$.

If $p = 2$, then we can take $(r, s) = (0, 1)$ and $r = p - 2$. This completes the proof. □

By Lemma 2.1 and the proof, we have $T(p, q) \to T(p - 2, q - 2k)$ by a self-fusion. Then, as the next step, we perform a self-fusion on $T(p - 2, q - 2k)$. Then we have the following Lemma by the arguments similar to the proof of Lemma 2.1:

Lemma 2.2 *A necessary and sufficient condition to get the sequence $T(p, q) \to T(p - 2, q - 2k) \to T(p - 4, q - 4k) \to \cdots$ is $q \equiv \pm 1 \pmod p$.*

By this lemma, we complete the proof of the first half of Proposition 1.4.

By using the sequence in Lemma 2.2, we calculate the number of times to get the trivial knot.

Suppose p is odd, and put $p = 2n + 1$ for some $n > 0$. Then the length of the sequence $p \to p - 2 \to p - 4 \to \cdots \to 3 \to 1$ is $n = \frac{p-1}{2}$. On the other hand, $q = pk \pm 1 = (2n + 1)k \pm 1$ ($\because q \equiv \pm 1 \pmod p$). Hence the sequence for q is as follows:

$$(2n + 1)k \pm 1 \to (2n - 1)k \pm 1 \to (2n - 3)k \pm 1 \to \cdots \to 3k \pm 1 \to k \pm 1.$$

Then to get the trivial knot, we need the length $n = \frac{p-1}{2}$ because $3k \pm 1 > 1$ ($\because k > 0$).

Next suppose p is even, and put $p = 2n$ for some $n > 0$. Then the length of the sequence $p \to p - 2 \to p - 4 \to \cdots \to 2 \to 0$ is $n = \frac{p}{2}$. On the other hand, $q = pk \pm 1 = 2nk \pm 1$ ($\because q \equiv \pm 1 \pmod p$). Hence the sequence for q is as follows:

$$2nk \pm 1 \to (2n-2)k \pm 1 \to (2n-4)k \pm 1 \to \cdots \to 2k \pm 1 \to \pm 1.$$

If $2k \pm 1 = 1$, then $k = 1$ ($\because k > 0$) and $q = p - 1$. In this case, the length of the sequence is $n - 1 = \frac{p-2}{2}$. Otherwise the length is $n = \frac{p}{2}$. This completes the proof of Proposition 1.4. $\square$

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